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From integrable to superintegrable

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Abstract
It is shown that the presence of multiple eigenvalue parameters may result in the superintegrability of the restricted soliton flows. The restricted AKNS flow, the Neumann system and the restricted mKdV flow with an eigenvalue parameter whose multiplicity is greater than two are demonstrated to be superintegrable.

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1. Introduction

In recent years, there has been growing interest in the study of superintegrable Hamiltonian systems. Recall that a classical Hamiltonian system on a $2N$-dimensional phase space is said to be superintegrable if it admits $N + k$, $1 \leqslant k \leqslant N - 1$ functionally independent and globally defined integrals of motion, N of them in involution pairwise. In particular, if the number of integrals takes the value $2N - 1$, then the system is called maximally superintegrable. The well-known elementary examples are the isotropic harmonic oscillator, the Kepler system and the Calogero–Moser system. A considerable effort has recently been devoted to a systematic search for superintegrable systems as well as the analysis to the structures of superintegrabilities of these models [1–5]. In particular, very recently Ballesteros and Herranz have proposed an approach to construct integrals for superintegrable systems on N-dimensional space of constant curvature which explains well the superintegrabilities of a class of superintegrable systems [4]. Yet, the general mechanism of superintegrability has remained unclear.

The aim of the present work is to uncover a mechanism of superintegrability for a kind of finite-dimensional integrable Hamiltonian systems called the restricted soliton flows. This kind of integrable systems contains a large number of important physical systems such as the Neumann system, the Garnier system, the Hénon–Heiles system and the geodesic flow equation on the ellipsoid. They can be obtained from (1+1)-dimensional soliton equations through nonlinearizations of spectral problems of soliton equations [6–9] and can be used to

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construct the exact solutions or the numeric solutions of the soliton equations [10–13]. In this paper, we confine our attention to the restricted flows of the soliton equations which relate to 2×2 matrix spectral problems. Viewing from Lax matrices, these systems are the generalized classic Gaudin model [14]

$$L(\lambda) = L_0(\lambda) + \Gamma_1, \qquad \Gamma_1 = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} q_j p_j & -q_j^2 \\ p_j^2 & -q_j p_j \end{pmatrix},$$

or the generalized Gaudin magnet with boundary [15],

$$L(\lambda) = L_0(\lambda) + \Gamma_2, \qquad \Gamma_2 = \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda q_j p_j & -\lambda_j q_j^2 \\ \lambda_j p_j^2 & -\lambda q_j p_j \end{pmatrix},$$

where $L_0(\lambda)$ is a traceless 2×2 matrix whose entries are polynomials of λ or λ^{-1}. One feature of them is that they depend on N parameters $\lambda_1, \ldots, \lambda_N$, called eigenvalue parameters. In what follows, we take three examples to demonstrate that superintegrability will occur provided that the degree of a multiple eigenvalue parameter is greater than two.

2. Some superintegrable Hamiltonian systems

Let us first consider the following restricted AKNS flow [7, 16]:

$$\begin{cases} q_x = -\Lambda q + \langle q, q\rangle p, \\ p_x = -\langle p, p\rangle q + \Lambda p, \end{cases} \tag{1}$$

where $q = (q_1, \ldots, q_N)^T$, $p = (p_1, \ldots, p_N)^T$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $\lambda_1, \ldots, \lambda_N$ are N arbitrary eigenvalue parameters.

It is a Hamiltonian system over the standard symplectic space $\left(R^{2N}, \sum_{j=1}^{N} \mathrm{d}p_j \wedge \mathrm{d}q_j\right)$

$$q_x = \{q, H\} \equiv \frac{\partial H}{\partial p}, \qquad p_x = \{p, H\} \equiv -\frac{\partial H}{\partial q}, \tag{2}$$

where

$$H = -\langle \Lambda q, p\rangle + \tfrac{1}{2}\langle q, q\rangle\langle p, p\rangle,$$

and allows the Lax representation

$$\frac{\mathrm{d}}{\mathrm{d}x} L(\lambda) = [U(\lambda), L(\lambda)], \tag{3}$$

where

$$L(\lambda) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} -q_j p_j & q_j^2 \\ -p_j^2 & q_j p_j \end{pmatrix} \tag{4}$$

and

$$U(\lambda) = \begin{pmatrix} -\lambda & \langle q, q\rangle \\ -\langle p, p\rangle & \lambda \end{pmatrix}.$$

It is easy to check that the Lax matrix (4) satisfies an r-matrix relation. Hence, $\det L(\lambda)$ is a generating function of integral of motion. In particular, when all parameters $\lambda_1, \ldots, \lambda_N$ are distinct, we arrive at

$$\det L(\lambda) = -1 - \sum_{j=1}^{N} \frac{I_j}{\lambda - \lambda_j},$$

where

$$I_j = 2q_j p_j - \sum_{k=1,k\neq j}^{N} \frac{B_{jk}^2}{\lambda_j - \lambda_k}, \qquad 1 \leqslant j \leqslant N \tag{5}$$

and

$$B_{jk} = q_j p_k - q_k p_j, \qquad 1 \leqslant j, \quad k \leqslant N.$$

$I_1, \ldots, I_N$ are N functionally independent and conversed integrals of motion to support the complete integrability of the Hamiltonian system (2). The relation between Hamiltonian H and $\lambda_1, \ldots, \lambda_N$ is

$$H = -\frac{1}{2}\sum_{j=1}^{N} \lambda_j I_j + \frac{1}{8}\left(\sum_{j=1}^{N} I_j\right)^2.$$

Following [17] we now consider the case of multiple eigenvalue parameters. For the sake of simplicity, we only discuss the case that $\lambda_1 = \cdots = \lambda_r = \alpha, \lambda_{r+1}, \ldots, \lambda_N$ are different, where $r \geqslant 2$. In this case, it can be checked that the Hamiltonian formula (2) and Lax representation (3) hold true. But the integrals $I_1, \ldots, I_r$ make no sense. However, we can construct $N + r - 2$ functionally independent integrals of motion as follows. First of all, we find that

$$\begin{aligned} K &= I_1 + \cdots + I_r \\ &= 2\sum_{j=1}^{r} q_j p_j - \sum_{k=r+1}^{N} \frac{1}{\alpha - \lambda_k}\left(B_{1k}^2 + \cdots + B_{rk}^2\right) \end{aligned} \tag{6}$$

exists in spite of $\lambda_1 = \cdots = \lambda_r = \alpha$. Moreover, K and $I_{r+1}, \ldots, I_N$ are in involution mutually.

Second, we observe that

$$\lim_{\lambda_j \to \lambda_k} [(\lambda_j - \lambda_k)(I_j - I_k)] = -2B_{jk}^2.$$

With it we define two groups of functions

$$K_{(k)} = \sum_{1\leqslant i<j\leqslant k} B_{ij}^2, \qquad 2 \leqslant k \leqslant r, \tag{7}$$

$$K^{(k)} = \sum_{r-k+1\leqslant i<j\leqslant r} B_{ij}^2, \qquad 2 \leqslant k \leqslant r. \tag{8}$$

A straightforward verification shows that

$$\left\{B_{jk}^2, B_{jn}^2\right\} = 4B_{jk}B_{jn}B_{kn}. \tag{9}$$

Therefore we have

$$\left\{B_{jk}^2, I_s\right\} = \frac{1}{\alpha - \lambda_s}\left\{B_{jk}^2, B_{sj}^2 + B_{sk}^2\right\} = 0, \qquad s \geqslant r+1, \quad j,k = 1, \ldots, r$$

and

$$\{K_{(j)}, K_{(l)}\} = 0, \qquad \{K^{(j)}, K^{(l)}\} = 0, \qquad 2 \leqslant j, \quad l \leqslant N.$$

As for the Poisson bracket $\{K_{(i)}, K^{(j)}\}$, we know that if $1 \leqslant i \leqslant \left[\frac{r}{2}\right], \left[\frac{r}{2}\right] + 1 \leqslant j \leqslant r$, then $\{K_{(i)}, K^{(j)}\} = 0$, and otherwise $\{K_{(i)}, K^{(j)}\}$ does not vanish, by a direct calculation.

Therefore, we arrive at two groups of independent integrals of motion in involution $\{K, K_{(2)}, \ldots, K_{(r)}, I_{r+1}, \ldots, I_N\}$ and $\{K, K^{(2)}, \ldots, K^{(r)}, I_{r+1}, \ldots, I_N\}$.

We remark that $K^{(r)} = K_{(r)}$ and

$$H = -\frac{1}{2}\left(\alpha K + K_{(r)} + \sum_{j=r+1}^{N} \lambda_j I_j\right) + \frac{1}{8}\left(K + \sum_{j=r+1}^{N} I_j\right)^2.$$

Consequently, if $r = 2$, due to $K_{(2)} = K^{(2)}$, we only get N independent integrals of motion and the system (2) is completely integrable. However, if $r \geqslant 3$ we have additional $r - 2$ integrals of motion besides the independent integrals $\{K, K_{(2)}, \ldots, K_{(r)}, I_{r+1}, \ldots, I_N\}$. Also we can show that $K, K_{(2)}, \ldots, K_{(r)}, K^{(2)}, \ldots, K^{(r-1)}, I_{r+1}, \ldots, I_N$ are functionally independent by direct calculations (a proof for $3 \leqslant r \leqslant 8$ is given in the appendix). Therefore, in this case the system (2) is not only completely integrable but also superintegrable.

Next we consider the cerebrated Neumann system [18]

$$q_{xx} + (\langle q_x, q_x\rangle - \langle \Lambda q, q\rangle)q + \Lambda q = 0, \qquad \langle q, q\rangle = 1,$$

or equivalently

$$\begin{cases} q_x = p, \\ p_x = -\Lambda q - (\langle p, p\rangle - \langle \Lambda q, q\rangle)q, \\ \langle q, q\rangle = 1, \langle q, p\rangle = 0, \end{cases} \tag{10}$$

which describes the motion of a particle constrained on the sphere S^{N-1} in N-dimensional space submitted to harmonic forces. It is a restricted KdV flow [8] and admits the following Lax representation:

$$\frac{\mathrm{d}}{\mathrm{d}x} L(\lambda) = [U(\lambda), L(\lambda)] \tag{11}$$

with

$$L(\lambda) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} q_j p_j & -q_j^2 \\ p_j^2 & -q_j p_j \end{pmatrix} \tag{12}$$

and

$$U(\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda - \langle p, p\rangle + \langle \Lambda q, q\rangle & 0 \end{pmatrix}. \tag{13}$$

This can be written as a Hamiltonian form

$$q_x = \{q, H\}_D, \ p_x = \{p, H\}_D \tag{14}$$

with

$$H = \tfrac{1}{2}(\langle p, p\rangle + \langle \Lambda q, q\rangle), \tag{15}$$

where $\{\cdot, \cdot\}_D$ is the Dirac bracket over the sphere bundle TS^{N-1}

$$TS^{N-1} = \{(q, p) \in R^{2N} | \Omega_1 \equiv \langle q, q\rangle - 1 = 0, \Omega_2 \equiv \langle q, p\rangle = 0\}, \tag{16}$$

i.e.

$$\{f, g\}_D = \{f, g\} + \tfrac{1}{2}\{f, \Omega_1\}\{\Omega_2, g\} - \tfrac{1}{2}\{f, \Omega_2\}\{\Omega_1, g\}.$$

It is well known that the Lax matrix $L(\lambda)$ satisfies an r-matrix relation and thus $\det L(\lambda)$ is a generating function of integrals of motion. Explicitly, if $\lambda_1, \ldots, \lambda_N$ are distinct we get

$$\det L(\lambda) = \sum_{j=1}^{N} \frac{I_j}{\lambda - \lambda_j},$$

where

$$I_j = q_j^2 + \sum_{k=1,k\neq j}^{N} \frac{B_{jk}^2}{\lambda_j - \lambda_k}, \qquad 1 \leqslant j \leqslant N.$$

In particular, $H = \frac{1}{2}\sum_{j=1}^{N} \lambda_j I_j$. Therefore, the Neumann system (10) is completely integrable when $\lambda_1, \ldots, \lambda_N$ are distinct.

Again when $\lambda_1 = \cdots = \lambda_r = \alpha, \lambda_{r+1}, \ldots, \lambda_N$ are different, we have

$$K = \sum_{j=1}^{r} q_j^2 + \sum_{k=r+1}^{N} \frac{1}{\alpha - \lambda_k}\left(B_{1k}^2 + \cdots + B_{rk}^2\right),$$

and $K_2, \ldots, K_r$ defined as before.

$\{K, K_{(2)}, \ldots, K_{(r)}, I_{r+1}, \ldots, I_N\}$ and $\{K, K^{(2)}, \ldots, K^{(r)}, I_{r+1}, \ldots, I_N\}$ are two groups of integrals of motion in involution pairwise. On the other hand, on TS^{N-1} we have

$$K + \sum_{j=r+1}^{N} I_j = \langle q, q\rangle = 1$$

and

$$H = \frac{1}{2}\left(\alpha K + K_{(r)} + \sum_{j=r+1}^{N} \lambda_j I_j\right).$$

Therefore, we only have $N + r - 3$ functionally independent integrals of motion $K, K_{(2)}, \ldots, K_{(r)}, I_{r+1}, \ldots, I_{N-1}, K^{(2)}, \ldots, K^{(r-1)}$. Hence, as $r \geqslant 3$ the Neumann system is not only complete integrable but also superintegrable.

The third example is the restricted mKdV flow [19]

$$\begin{cases} q_x = -\langle q, p\rangle q + \Lambda p, \\ p_x = \Lambda q + \langle q, p\rangle p, \end{cases} \tag{17}$$

which is a Hamiltonian system

$$q_{k,x} = \frac{\partial H}{\partial p_k}, \qquad p_{k,x} = -\frac{\partial H}{\partial q_k}, \qquad 1 \leqslant k \leqslant N,$$

where

$$H = -\tfrac{1}{2}(\langle q, p\rangle^2 - \langle \Lambda p, p\rangle + \langle \Lambda q, q\rangle).$$

It allows the following Lax representation:

$$\frac{\mathrm{d}}{\mathrm{d}x} L(\lambda) = [U(\lambda), L(\lambda)], \tag{18}$$

with

$$L(\lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \sum_{k=1}^{N} \frac{1}{\lambda^2 - \lambda_k^2} \begin{pmatrix} \lambda q_k p_k & -\lambda_k q_k^2 \\ \lambda_k p_k^2 & -\lambda q_k p_k \end{pmatrix} \tag{19}$$

and

$$U(\lambda) = \begin{pmatrix} -\langle q, p\rangle & \lambda \\ \lambda & \langle q, p\rangle \end{pmatrix}. \tag{20}$$

$\det L(\lambda)$ is a generating function of integrals of motion. Explicitly, when $\lambda_1, \ldots, \lambda_N$ are distinct we arrive at

$$\det L(\lambda) = -1 + \sum_{j=1}^{N} \frac{I_j}{\lambda^2 - \lambda_j^2},$$

where

$$I_j = \lambda_j p_j^2 - \lambda_j q_j^2 - q_j^2 p_j^2 + \frac{1}{2}\lambda_j \sum_{k=1,k\neq j}^{N} \left(\frac{B_{jk}^2}{\lambda_j - \lambda_k} - \frac{P_{jk}^2}{\lambda_j + \lambda_k} \right),$$

where $P_{jk} = q_j p_k + q_k p_j$, $1 \leqslant j, k \leqslant N$.

$$H = \frac{1}{2}\sum_{j=1}^{N} I_j.$$

Again, when $\lambda_1 = \cdots = \lambda_r = \alpha, \lambda_{r+1}, \ldots, \lambda_N$ are distinct, we have

$$K = \sum_{j=1}^{r} I_j$$

$$= \sum_{j=1}^{r} \left(\lambda_j p_j^2 - \lambda_j q_j^2 - q_j^2 p_j^2\right) + \frac{1}{2} \sum_{1\leqslant i<j\leqslant r} \left(B_{ij}^2 - P_{ij}^2\right) + \frac{1}{2}\sum_{j=1}^{r}\sum_{k=r+1}^{N} \left(\frac{\alpha B_{jk}^2}{\alpha - \lambda_k} - \frac{\alpha P_{jk}^2}{\alpha + \lambda_k} \right)$$

and

$$\lim_{\lambda_j \to \lambda_k} [(\lambda_j - \lambda_k)(I_j - I_k)] = \lambda_k B_{jk}^2, \qquad H = \frac{1}{2}\left(K + \sum_{j=r+1}^{N} I_j\right).$$

Integrals of motion $K_{(k)}, K^{(k)}$ defined as before. Finally, $K, K_{(2)}, \ldots, K_{(r)}, I_{r+1}, \ldots, I_N$, $K^{(2)}, \ldots, K^{(r-1)}$ are functionally independent integrals of motion of the restricted mKdV flow. When $r \geqslant 3$, the restricted mKdV flow is superintegrable.

3. Conclusion and discussion

It has been shown that the presence of multiple parameters may result in superintegrability of the restricted soliton flows. We take the restricted AKNS flow, the Neumann system and the restricted mKdV flow as illustrative examples. We remark if all eigenvalue parameters are identical then we recover the result of [4].

We note that recently there has been some work done on the integrable systems with multiple eigenvalue parameters [20, 21]. In particular, in [20] Vuk has observed that the confluent Neumann system with $r > 2$ identical eigenvalue parameters is superintegrable and showed by a counterexample that if only two of the eigenvalue parameters coincide the confluent Neumann systems are not superintegrable. Therefore, maybe it is necessary for the superintegrability of the restricted soliton flow that the eigenvalue parameter multiplicity is greater than two.

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Appendix. On the functionally independence of first integrals

We first show the functionally independence of first integrals of the restricted AKNS flow (1) with multiple eigenvalue parameters. More explicitly, we show that the matrix

$$J(r) = \frac{\partial(K, K_{(2)}, \ldots, K_{(r)}, K^{(r-1)}, \ldots, K^{(2)}, I_{r+1}, \ldots, I_N)}{\partial(p_1, \ldots, p_{r-2}, q_1, \ldots, q_r, q_{r+1}, \ldots, q_N)} \tag{A.1}$$

is nongenerate on the domain
$M=\{(q,p)\in R^{2N}\,|\,p_{2j+1}=q_{2k}=p_l=0,\ 1\leqslant 2j+1\leqslant r-2,\ 1\leqslant 2k\leqslant r,\ r+1\leqslant l\leqslant N\}$,
where I_j, K, K_k and $K^{(k)}$ are defined by (5)–(8).

To this end, introduce notation

$$\tilde{K}=K+\sum_{j=r+1}^{N}I_j=2\sum_{k=1}^{N}q_kp_k,$$

$$K(k)=K_{(k)}-K_{(k-1)}=\sum_{j=1}^{k-1}B_{jk}^2,\qquad k=3,\ldots,r,$$

$$f(m)=K^{(m)}-K^{(m-1)}=\sum_{j=m+1}^{r}B_{mj}^2,\qquad m=3,\ldots,r-1$$

and

$$K(2)=K_{(2)}\equiv B_{12}^2,\qquad f(2)=K^{(2)}\equiv\sum_{j=3}^{r}B_{2j}^2.$$

By properties of determinant it is easy to see that

$$\det(J(r))=\det(\tilde{J}(r))$$

where

$$\tilde{J}(r)=\frac{\partial(\tilde{K},K(2),\ldots,K(r),f(2),\ldots,f(r-1),I_{r+1},\ldots,I_N)}{\partial(p_1,\ldots,p_{r-2},q_1,\ldots,q_r,q_{r+1},\ldots,q_N)}.$$

Observe that

$$\det(\tilde{J}(r))=\begin{vmatrix}\tilde{K}_L & \tilde{K}_R\\ \frac{\partial(K(2),\ldots,K(r),f(2),\ldots,f(r-1))}{\partial(p_1,\ldots,p_{r-2},q_1,\ldots,q_r)} & 0\\ \frac{\partial(I_{r+1},\ldots,I_N)}{\partial(p_1,\ldots,p_{r-2},q_1,\ldots,q_r)} & \frac{\partial(I_{r+1},\ldots,I_N)}{\partial(q_{r+1},\ldots,q_N)}\end{vmatrix},$$

where $\tilde{K}_L=2(q_1,\ldots,q_{r-2},p_1,\ldots,p_r)$ and $\tilde{K}_R=2(p_{r+1},\ldots,p_N)$ are two row vectors.

Since

$$\frac{\partial I_{r+j}}{\partial q_{r+k}}=2p_{r+j}\delta_{jk}-\frac{2\left(\sum_{l=1}^{r}B_{r+j,l}p_l\right)}{\lambda_{r+j}-\alpha}\delta_{jk}$$
$$-\sum_{k=1,k\neq j}^{N}\frac{2}{\lambda_{r+j}-\lambda_{r+k}}B_{r+j,r+k}p_{r+k},\qquad 1\leqslant j,\quad k\leqslant N-r,$$

we have

$$\left.\frac{\partial(I_{r+1},\ldots,I_N)}{\partial(q_{r+1},\ldots,q_N)}\right|_{p_{r+1}=\cdots=p_N=0}=2\left(\sum_{j=1}^{r}p_j^2\right)\mathrm{diag}\left(\frac{q_{r+1}}{\alpha-\lambda_{r+1}},\ldots,\frac{q_N}{\alpha-\lambda_N}\right)$$

and thus

$$\det(\tilde{J}(r))|_{p_{r+1}=\cdots=p_N=0}$$
$$=\begin{vmatrix}\tilde{K}_L & 0\\ \frac{\partial(K(2),K(3),\ldots,K(r),f(2),\ldots,f(r-1))}{\partial(p_1,\ldots,p_{r-2},q_1,\ldots,q_r)} & 0\\ \frac{\partial(I_{r+1},\ldots,I_N)}{\partial(p_1,\ldots,p_{r-2},q_1,\ldots,q_r)} & \left.\frac{\partial(I_{r+1},\ldots,I_N)}{\partial(q_{r+1},\ldots,q_N)}\right|_{p_{r+1}=\cdots=p_N=0}\end{vmatrix}$$
$$=2^{N-r}\sum_{j=1}^{r}p_j^2\prod_{k=r+1}^{N}\frac{q_k}{\alpha-\lambda_k}D_r$$

where

$$D_r = \left|\frac{\tilde{K}_L}{\frac{\partial(K(2),K(3),\dots,K(r),f(2),\dots,f(r-1))}{\partial(p_1,\dots,p_{r-2},q_1,\dots,q_r)}}\right| = \left|\frac{\partial(S,K(2),K(3),\dots,K(r),f(2),\dots,f(r-1))}{\partial(p_1,\dots,p_{r-2},q_1,\dots,q_r)}\right|$$

only depends on $p_1,\dots,p_r,q_1,\dots,q_r$ and $S = 2\sum_{j=1}^{r} q_j p_j$.

Therefore the functionally independence of $K, K_2,\dots,K_r,K^{(2)},\dots,K^{(r-1)},I_{r+1},\dots,I_N$ is equivalent to $D_r \neq 0$ over some domain. We can readily obtain that

$$D_3(p_1 = q_2 = 0) = 16q_1^2q_3^2p_2^3p_3(p_2^2+p_3^2),$$
$$D_4(p_1 = q_2 = q_4 = 0) = 64q_1^4q_3^3p_2^2p_3^2p_4^3(p_2^2+p_3^2+p_4^2).$$

For an explicit integer r $(r \geqslant 3)$ we can work out D_r with the following Maple programme:

```
r :=;
with linalg
B := (i, j)-> q[i] * p[j] - q[j] * p[i];
K := (k)-> sum(B(j, k)^2, j = 1, ..., k - 1);
f := (m)-> sum(B(m, s)^2, s = m + 1, ..., r);
S := 2 * sum(q[j] * p[j], j = 1, ..., r);
g:= proc (z) if mod(z, 2) = 0 then p[z] else 0 end if end proc
 h:= proc (z) if mod(z, 2) = 1 then q[z] else 0 end if end proc
with(VectorCalculus)
A, d := Jacobian([S, seq(K(e), e=2..r), seq(f(e), e=2..r - 1)], [seq(p[e], e=1..r - 2),
seq(q[e], e = 1..r)] = [seq(g(z), z = 1..r - 2), seq(h(z), z = 1..r)], 'determinant');
simplify(d, size).
```

In particular, we have

$$D_5(p_1 = p_3 = p_2 = q_4 = 0) = -256q_1^2q_3^2q_5^4p_2^4p_4^3p_5^3(q_1^2+q_3^2)(p_2^2+p_4^2+p_5^2),$$

$$D_6(p_1 = p_3 = q_2 = q_4 = q_6 = 0) = -1024q_1^2q_3^2q_5^5p_2^4p_4^2p_5^4p_6^3(q_1^2+q_3^2)^2(p_2^2+p_4^2+p_5^2+p_6^2),$$

$$D_7(p_1 = p_3 = p_5 = q_2 = q_4 = q_6 = 0) = 4096q_1^2q_3^2q_5^2q_7^6p_2^4p_4^2p_6^3p_7^5(q_1^2+q_3^2)$$
$$(q_1^2+q_3^2+q_5^2)(p_2^2+p_4^2)(p_2^2+p_4^2+p_6^2+p_7^2),$$

$$D_8(p_1 = p_3 = p_5 = q_2 = q_4 = q_6 = q_8 = 0) = 16384q_1^2q_3^2q_5^2q_7^7p_2^4p_4^2p_6^2p_7^6p_8^3(q_1^2+q_3^2)$$
$$(q_1^2+q_3^2+q_5^2)^2(p_2^2+p_4^2)(p_2^2+p_4^2+p_6^2+p_7^2+p_8^2).$$

These express that $K, K_2,\dots,K_r,K^{(r-1)},\dots,K^{(2)},I_{r+1},\dots,I_N$ are functionally independent when $3 \leqslant r \leqslant 8$ over M.

The same technique can be used in the proofs of the functionally independence of first integrals of the Neumann systems and the restricted mKdV flow with multiple eigenvalue parameters.

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